Phase 1 of the simplex algorithm

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Introduction

Reference problem

$$\begin{aligned} \max\ & c^T x \\ Ax &= b \\ x &\geq 0 \end{aligned} \qquad \text{(LP-S)}$$

Assumption: $b \geq 0$

↳ without loss of generality (if $b_i < 0$ multiply by -1 both sides of $a_{i,\cdot}\, x = b_i$)

Problem: Compute a BFS or conclude (LP-S) is infeasible

Key trick

Build the auxiliary problem

$$\min_{x,y} \sum_{i=1}^{m} y_i \qquad \text{(LP-AUX)}$$

$$Ax + y = b$$

$$x \geq 0, \; y \geq 0$$

- $y \in \mathbb{R}^m$ are called artificial variables
- In matrix form we have

$$\tilde{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} A & I \end{bmatrix} \longrightarrow \min \; [\overbrace{0 \ldots 0}^{n \text{ zeros}} \; \overbrace{1 \ldots 1}^{m \text{ ones}}] \; \tilde{x}$$

$$\tilde{A}x = b$$

$$\tilde{x} \geq 0$$

Properties of the auxiliary problem

$$\min_{x,y} \sum_{i=1}^{m} y_i \qquad \text{(LP-AUX)}$$

$$Ax + y = b$$

$$x \geq 0,\ y \geq 0$$

- (LP-AUX) is always feasible because $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$ is a BFS (the one associated to the basis $B = I$ contained in $\hat{A}$)
- Let z^*_{AUX} be the optimal cost of (LP-AUX) and x^*, y^* be optimizers

$$z^*_{AUX} = 0 \iff \sum_{i=1}^{M} y_i^* = 0 \iff y^* = 0 \iff \begin{cases} Ax^* = b \\ x^* \geq 0 \end{cases} \iff$$ x^* is a feasible solution to (LP-S)

Thm. If $\tilde{x}^T = [x^* \; 0]$ is an optimal solution to (LP-AUX), then x^* is a BFS to (LP-S)

Tableau form of phase 1

- Initial tableau for (LP-AUX). Assumption: $b \geq 0$

$$
\begin{array}{l|c|ccc|ccc}
 & & x_1 & \dots & x_n & y_1 & \dots & y_m \\
\text{row } 0 \rightarrow & 0 & 0 & \dots & 0 & 1 & \dots & 1 \\
\hline
\text{row } 1 \rightarrow & & & & & & & \\
\vdots & b & & A & & & I & \\
\text{row } m \rightarrow & & & & & & & \\
\end{array}
$$

In order to put it in canonical form w.r.t. the variables $y_1, \dots, y_m$, subtract rows $1, \dots, m$ from row 0 (it is equivalent to pivoting on all 1's on the diagonal of I)

- How to get the initial tableau of phase 2 from the final tableau of phase 1. Only two cases are possible

- Case a : all y_i are NBVs

Let $x_{i_1}, \dots, x_{i_m}$ be BVs

$$
\begin{array}{c|c|ccc:ccc}
 & & x_1 & \cdots & x_n & y_1 & \cdots & y_m \\
 & 0 & & \bar{c} & & & * & \\
\hline
x_{i_1} & & & & & & & \\
\vdots & \bar{b} & & \bar{A} & & & * & \\
x_{i_m} & & & & & & &
\end{array}
\xrightarrow[\text{for phase 2}]{\text{Initial tableau}}
\begin{array}{c|ccc}
 & x_1 & \cdots & x_n \\
0 & & c & \\
\hline
\bar{b} & & \bar{A} &
\end{array}
$$

uninteresting entries

cost of (LP-S)

Then we put the tableau in canonical form w.r.t. $x_{i_1}, \dots x_{i_m}$ through suitable pivot operations

- Case b: a variable y_h is basic. Let the associated row be the i-th one

$$
\begin{array}{c|c|ccc|ccccc}
 & 0 & x_1 & \cdots & x_n & y_1 & \cdots & y_h & \cdots & y_m \\
\hline
\vdots & & & & & & & 0 & & \\
 & & & & & & & \vdots & & \\
 & & & & & & & 0 & & \\
y_h & 0 & \bar{a}_{i,1} & \cdots & \bar{a}_{i,n} & & \cdots & 1 & \cdots & \\
 & & & & & & & 0 & & \\
\vdots & & & & & & & \vdots & &
\end{array}
$$

↳ the BFS must be degenerate! Look for another representation of the same vertex without auxiliary BVs

- If some $\bar{a}_{i,j}$ is nonzero, perform a pivot on $\bar{a}_{i,j}$

  ↳ y_h leaves the basis and the cost remains 0

- If all $\bar{a}_{i,j}$ are zero, the i-th row can be removed from the tableau
  ↳ rank(A) was not maximal and some row of A could have been removed from the beginning

After these operations, proceed as in Case a.

Ex.

$$\min \; x_1 + x_3$$
$$x_1 + 2x_2 \leq 5$$
$$x_2 + 2x_3 = 6$$
$$x_1, x_2, x_3 \geq 0$$

Solve the LP problem running the simplex algorithm

Put the LP in standard form

$$\min \; x_1 + x_3$$
$$x_1 + 2x_2 + x_4 = 5$$
$$x_2 + 2x_3 = 6$$
$$x_1, x_2, x_3, x_4 \geq 0$$

slack variable

(LP-S)

Verify that $b \geq 0$. $b = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ OK!

Run simplex phase 1

- Build the auxiliary problem

the auxiliary pbl is always a "min"

$$\min \; y_1 + y_2$$

$$x_1 + 2x_2 + x_4 + y_1 = 5$$

$$x_2 + 2x_3 + y_2 = 6$$

$$x_1, \ldots, x_4 \geq 0, \quad y_1, y_2 \geq 0$$

→ auxiliary vars

- Initial Tableau

row 0 →

	x_1	x_2	x_3	x_4	y_1	y_2
0	0	0	0	0	1	1
5	1	2	0	1	1	0
6	0	1	2	0	0	1

Pivoting on these elements = subtract rows 1 and 2 from row 0

- Put the tableau in standard form w.r.t. variables y_1 and y_2

- Run phase 2 as a subroutine of phase 1

$$
\begin{array}{cc|cccccc}
 & & x_1 & x_2 & x_3 & x_4 & y_1 & y_2 \\
 & -11 & -1 & -3 & -2 & -1 & 0 & 0 \\
\hline
y_1 & 5 & 1 & 2 & 0 & 1 & 1 & 0 \\
y_2 & 6 & 0 & 1 & 2 & 0 & 0 & 1
\end{array}
\qquad
\begin{array}{l}
-\,[-5 \;\; -1 \;\; -2 \;\; 0 \;\; -1 \;\; -1 \;\; 0] \\
\\
-\,0
\end{array}
$$

$r_F \not\geq 0 \longrightarrow$ Bland's rule: x_1 enters the basis.

Ratios: $5/1 \to y_1$ leaves the basis

$6/0$

$\mathrm{Aux} = [5 \;\; 1 \;\; 2 \;\; 0 \;\; 1 \;\; 1 \;\; 0]$

		x_1	x_2	x_3	x_4	y_1	y_2
	-6	0	-1	-2	0	1	0
x_1	5	1	2	0	1	1	0
y_2	6	0	1	2	0	0	1

$$-\left[-\frac{5}{2} \quad -\frac{1}{2} \quad -1 \quad 0 \quad -\frac{1}{2} \quad -\frac{1}{2} \quad 0\right]$$

$$-\left[\frac{5}{2} \quad \frac{1}{2} \quad 1 \quad 0 \quad \frac{1}{2} \quad \frac{1}{2} \quad 0\right]$$

$r_F \not\geq 0 \longrightarrow$ Bland's rule: x_2 enters the basis.

Ratios: $5/2 \rightarrow x_1$ leaves the basis

$6/1$

$$Aux = \left[\frac{5}{2} \quad \frac{1}{2} \quad 1 \quad 0 \quad \frac{1}{2} \quad \frac{1}{2} \quad 0\right]$$

$$
\begin{array}{cc|cccccc}
 & & x_1 & x_2 & x_3 & x_4 & y_1 & y_2 \\
 & -\frac{7}{2} & \frac{1}{2} & 0 & -2 & \frac{1}{2} & \frac{3}{2} & 0 \\
\hline
x_2 & \frac{5}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
y_2 & \frac{7}{2} & -\frac{1}{2} & 0 & 2 & -\frac{1}{2} & -\frac{1}{2} & 1
\end{array}
$$

$- \left[-\frac{7}{2} \quad \frac{1}{2} \quad 0 \quad -2 \quad \frac{1}{2} \quad \frac{1}{2} \quad -1 \right]$

$- \; 0$

$r_F \not\geq 0 \longrightarrow$ Bland's rule: x_3 enters the basis.

Ratios: $\cancel{\frac{5/2}{0}}$

$\frac{7/2}{2} \rightarrow y_2$ leaves the basis

$Aux = \left[\frac{7}{4} \quad -\frac{1}{4} \quad 0 \quad 1 \quad -\frac{1}{4} \quad -\frac{1}{4} \quad \frac{1}{2} \right]$

		x_1	x_2	x_3	x_4	y_1	y_2
	0	0	0	0	0	1	1
x_2	$\frac{5}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0
x_3	$\frac{7}{4}$	$-\frac{1}{4}$	0	1	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$

→ zero cost: (LP-S) is feasible

- Compute the initial tableau for phase 2

		x_1	x_2	x_3	x_4
	0	1	0	1	0
x_2	$\frac{5}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$
	$\frac{7}{4}$	$-\frac{1}{4}$	0	1	$-\frac{1}{4}$

→ cost of (LP-S)

→ Initial BVs : x_2 and x_3

Rmk: x_2 is already a dependent variable but x_3 is not

Run simplex phase 2

- Put the initial tableau in canonical form w.r.t. the initial basis

		x_1	x_2	x_3	x_4
	0	1	0	1	0
x_2	$\frac{5}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$
	$\frac{7}{4}$	$-\frac{1}{4}$	0	1	$-\frac{1}{4}$

→ It is enough to turn x_3 into a dependent variable by pivoting on the circled entry

↳ This amounts to subtract row 2 from row 0

$$\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ -\frac{7}{4} & \frac{5}{4} & 0 & 0 & \frac{1}{4} \\ \hline x_2 \quad \frac{5}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} \\ x_3 \quad \frac{7}{4} & -\frac{1}{4} & 0 & 1 & -\frac{1}{4} \end{array}$$

$r_F \geq 0 \longrightarrow$ optimal solution: STOP.

- Read the results from the final tableau

Optimal cost: $\frac{7}{4}$

Optimal BFS $x_B^* = \begin{bmatrix} x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 5/2 \\ 7/4 \end{bmatrix} \longrightarrow x^* = \begin{bmatrix} 0 \\ 5/2 \\ 7/4 \\ 0 \end{bmatrix}$

Degeneracy and finite termination of the simplex algorithm

Ex.

$$\begin{aligned} \max\ & 2x_1 + x_2 \\ & 3x_1 + x_2 + x_3 = 6 \\ & x_1 - x_2 + x_4 = 2 \\ & x_2 + x_5 = 3 \\ & x_1, \ldots, x_5 \geq 0 \end{aligned} \qquad \text{(LP-S)}$$

Assume in phase 2 we computes the tableau

		x_1	x_2	x_3	x_4	x_5
	-4	0	3	0	2	0
x_3	0	0	4	1	-3	0
x_1	2	1	-1	0	1	0
x_5	3	0	1	0	0	1

$$-\left[\begin{matrix} 0 & 0 & 3 & \frac{3}{4} & -\frac{9}{4} & 0 \end{matrix}\right]$$

$$-\left[\begin{matrix} 0 & 0 & -1 & -\frac{1}{4} & \frac{3}{4} & 0 \end{matrix}\right]$$

$$-\left[\begin{matrix} 0 & 0 & 1 & \frac{1}{4} & -\frac{3}{4} & 0 \end{matrix}\right]$$

$c_F \nleq 0 \rightarrow$ Bland's rule: x_2 enters the basis.

Ratios: $0/4 \longrightarrow$ x_3 leaves the basis but the cost cannot change

$2/-1$

$3/1$

$$AUX = \left[\begin{matrix} 0 & 0 & 1 & \frac{1}{4} & -\frac{3}{4} & 0 \end{matrix}\right]$$

		x_1	x_2	x_3	x_4	x_5
	-4	0	0	$-3/4$	$17/4$	0
x_3	0	0	1	$1/4$	$-3/4$	0
x_1	2	1	0	$1/4$	$1/4$	0
x_5	3	0	0	$-1/4$	$3/4$	1

↳ New BFS but same cost

Rmk.

- If a vertex is defined by more than n hyperplanes, there are multiple feasible basis corresponding to it
- On the example, further iterations of phase 2 will generate other vertices with higher cost

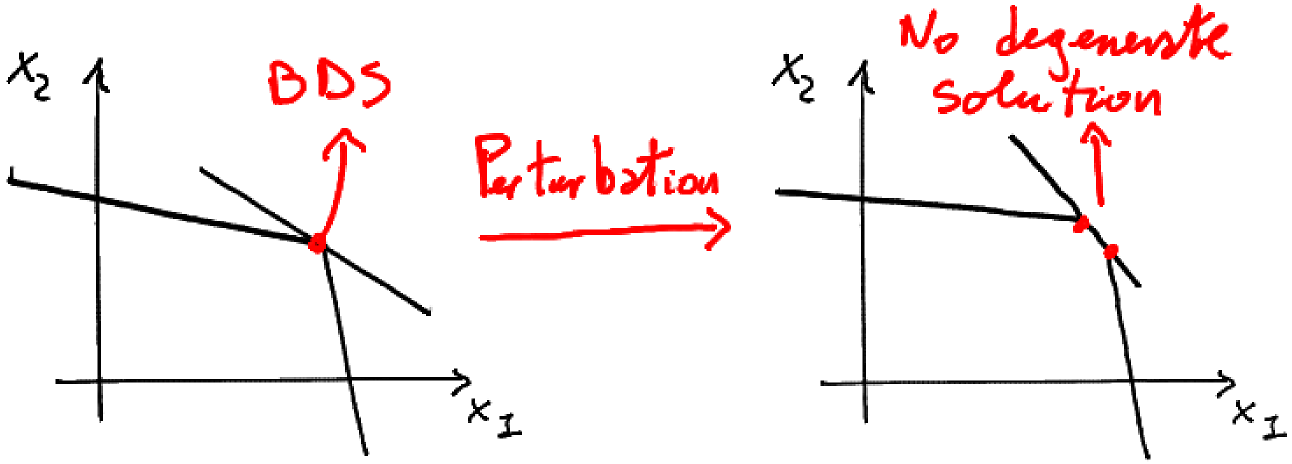
Pbl: How to guarantee, in general that the simplex algorithm does not cycle forever among degenerate basis?

Anti-cycling rules

1) Bland's rule

Thm. The simplex algorithm with Bland's rule ends in at most $\binom{n}{m}$ iterations.

2) Random "small" perturbations of constraints after detecting a cycle



↳ Pbl. Perturbations may compromise feasibility

↳ Several heuristics available ...